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LETTER TO THE EDITOR

Explicit wavefunctions for shape-invariant potentials by operator techniques

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Abstract. We obtain explicit expressions for the wavefunctions of all the known shape-invariant potentials by using the recently proposed operator method.

Recently, the familiar harmonic oscillator raising and lowering operator technique has been generalised to other potentials of physical interest (Gendenshtein 1983, Dutt *et al* 1986a, b). The generalisation, which in many respects is equivalent to Schrödinger's factorisation method (Schrödinger 1940, 1941, Infeld and Hull 1951), is based on two main concepts—supersymmetry and shape-invariant potentials. For quantum mechanical purposes, the main implication of supersymmetry is simply stated. Given any potential $V_-(x)$, supersymmetry allows one to construct a partner potential $V_+(x)$ with the same energy eigenvalues (except for the ground state) (Witten 1981, Cooper and Freedman 1983). Furthermore, if $V_-(x)$ and $V_+(x)$ have similar shapes, they are said to be 'shape invariant'. The concept was introduced five years ago by Gendenshtein (1983) who, in the same paper, calculated the energy eigenvalues of such potentials and pointed out that many of the solvable potentials (such as Coulomb, harmonic oscillator, Morse, Eckart, Pöschl–Teller, etc) are shape invariant (see Cooper *et al* (1987) for a detailed discussion regarding the connection between shape invariance and solvable potentials). This work was subsequently extended (Dutt *et al* 1986a, b) to obtain a formal operator expression for the bound-state wavefunctions in terms of the ground state in a manner analogous to the harmonic oscillator operator method.

Although the operator formalism provides an elegant way of writing the eigenfunctions, it is often desirable to have explicit expressions for them. The purpose of this letter is to obtain explicit wavefunctions for all known shape-invariant potentials by using the operator method.

In supersymmetric quantum mechanics, the supersymmetric partner potentials $V_\pm(x)$ are given by ($\hbar = 2m = 1$)

$$V_\pm(x) = W^2(x) \pm W'(x) \qquad W'(x) = \mathrm{d}W/\mathrm{d}x \tag{1}$$

where $W(x)$ is the superpotential. The Hamiltonians corresponding to these potentials can be written in a factorised form in terms of the operators A and A^+:

$$H_- = A^+A \qquad H_+ = AA^+ \qquad A^+ = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x). \tag{2}$$

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It may be noted here that $W(x)$ is simply related to the ground-state eigenfunction ψ_0 of H_-:

$$W(x) = -\psi_0'(x)/\psi_0(x) \qquad \psi_0(x) = \exp\left(-\int^x W(x)\,dx\right). \tag{3}$$

Now, if $\psi_n^{(-)}$ and $\psi_n^{(+)}$ denote the eigenfunctions of the Hamiltonians H_- and H_+ respectively with eigenvalues $E_n^{(-)}$ and $E_n^{(+)}$, then it is well known that (see, for example, Witten 1981)

$$E_n^{(+)} = E_{n+1}^{(-)} \tag{4}$$

$$\psi_n^{(+)} \propto A\psi_{n+1}^{(-)} \qquad \psi_{n+1}^{(-)} \propto A^+\psi_n^{(+)} \qquad n = 0, 1, 2, \ldots \tag{5}$$

i.e. the operator A (A^+) not only converts an eigenfunction of H_- (H_+) into an eigenfunction of H_+ (H_-) with the same energy, but it also destroys (creates) a node.

Let us now explain precisely what one means by shape-invariant potentials. If the pair of supersymmetric partner potentials $V_\pm(x)$ defined by (1) are similar in shape and differ only in the parameters which appear in them, then they are said to be shape invariant. More specifically, if $V_-(x; a_0)$ is any potential, its supersymmetric partner $V_+(x; a_0)$ must satisfy the requirement (Gendenshtein 1983)

$$V_+(x; a_0) = V_-(x; a_1) + R(a_1) \tag{6}$$

where a_0 is a set of parameters, a_1 is a function of a_0 ($a_1 = f(a_0)$, say) and the remainder $R(a_1)$ is independent of x. Using these ideas, the eigenstates of shape-invariant potentials can easily be obtained. In particular, one can show that the energy spectrum of H_- is given by (Gendenshtein 1983)

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k) \qquad E_0^{(-)} = 0 \qquad a_k = f^k(a_0) \tag{7}$$

while its unnormalised energy eigenfunctions are given by (Dutt *et al* 1986a, b)

$$\psi_n^{(-)}(x, a_0) = A^+(x, a_0)A^+(x, a_1)\ldots A^+(x, a_{n-1})\psi_0^{(-)}(x, a_n) \tag{8}$$

which is clearly a generalisation of the operator method of constructing the energy eigenfunctions for the harmonic oscillator potential problem.

Although (8) is an elegant formal expression, one often wants to see the explicit coordinate dependence of the bound-state wavefunctions. In order to do that, it is necessary to find the effect of repeated operations of A^+ (with different parameters a_0, a_1, etc) on the ground-state wavefunction ψ_0 and then recognise the result in terms of known functions. Instead of this lengthy procedure, it is far simpler to use the result

$$\psi_n(x; a_0) = A^+(x; a_0)\psi_{n-1}(x; a_1). \tag{9}$$

In (9) and all subsequent discussion, we only consider wavefunctions of $V_-(x)$, so the superscript $(-)$ has been suppressed for simplicity. We now rewrite (9) in a form so as to recognise recursion relations for known functions. To illustrate this procedure we first obtain explicit expressions for $\psi_n(x, a_0)$ for two potentials.

(1) Simple harmonic oscillator potential: $V_-(x; a_0) = \frac{1}{4}\omega^2(x - 2b/\omega)^2 - \frac{1}{2}\omega$. For this case

$$W(x; \omega) = x - 2b/\omega \qquad \psi_0(x; \omega) = \exp\left(-\frac{x^2}{2} + \frac{2b}{\omega}x\right). \tag{10}$$

Also the parameter $a_0 = \omega$ and all the shape-invariant partner potentials have the same parameter, i.e. $a_k = \omega$ $(k = 0, 1, 2, \ldots)$. Hence (9) yields

$$\psi_n(y;\,\omega) = (-\mathrm{d}/\mathrm{d}y + \tfrac{1}{2}\omega y)\psi_{n-1}(y;\,\omega) \tag{11}$$

where $y = x - 2b/\omega$. On defining

$$\psi_n(y;\,\omega) = \psi_0(y;\,\omega)R_n(y;\,\omega) \tag{12}$$

one finds that $R_n(y;\,\omega)$ satisfies

$$\sqrt{2n}\;R_n(y;\,\omega) = 2yR_{n-1}(y;\,\omega) - (\mathrm{d}/\mathrm{d}y)R_{n-1}(y,\,\omega). \tag{13}$$

On comparing with the recursion relation (A1) for the Hermite polynomials we see that $R_n\;(y;\,\omega)$ is proportional to $H_n(y)$. Hence the unnormalised wavefunctions for the one-dimensional harmonic oscillator problem are

$$\psi_n(y;\,\omega) \propto \exp(-y^2/2)H_n(y). \tag{14}$$

It may be added here that this example has been discussed in many textbooks (see, for example, Böhm 1979). We have still decided to give it so that it becomes easier to appreciate the application of (9) to other shape-invariant potentials. We now discuss a second example.

(ii) Pöschl–Teller potential:

$$V_-(x) = -(A+B)^2 + A(A-\alpha)\sec^2\alpha x + B(B-\alpha)\operatorname{cosec}^2\alpha x \quad (0 \leqslant \alpha x \leqslant \pi/2;\; A,\,B > 0).$$

Here

$$W(x) = A\tan\alpha x - B\cot\alpha x \qquad \psi_0(x) = (\sin\alpha x)^{B/\alpha}(\cos\alpha x)^{A/\alpha}. \tag{15}$$

This potential depends on two parameters $\{a_0\} = (A,\,B)$ and the first partner potential has parameters $\{a_1\} = (A+\alpha,\,B+\alpha)$. Hence in this case (9) takes the form

$$\psi_n(x;\,\{a_0\}) = (-\mathrm{d}/\mathrm{d}x + A\tan\alpha x - B\cot\alpha x)\psi_{n-1}(x;\,\{a_1\}). \tag{16}$$

As in the previous example we define a new variable

$$y = 1 - 2\sin^2\alpha x \tag{17}$$

and factor out the ground-state wavefunction

$$\psi_n(y,\,\{a_0\}) = \psi_0(y,\,\{a_0\})R_n(y,\,\{a_0\}). \tag{18}$$

Substituting into (16) and using (15) we then obtain

$$\begin{aligned}\frac{1}{\alpha}\,R_n(y;\,A,\,B) &\\ &= (1-y^2)\frac{\mathrm{d}}{\mathrm{d}y}R_{n-1}(y;\,A+\alpha,\,B+\alpha)\\ &\quad + \left[\frac{(A-B)}{\alpha} - \left(\frac{A+B}{\alpha}+1\right)y\right]R_{n-1}(y;\,A+\alpha,\,B+\alpha).\end{aligned} \tag{19}$$

On comparison with the recursion relation (A3b) we see that $R_n(y;\,A,\,B)$ is proportional to the Jacobi polynomial $P_n^{(\alpha,\beta)}(y)$ with

$$\alpha = B/\alpha - \tfrac{1}{2} \qquad \beta = A/\alpha - \tfrac{1}{2}. \tag{20}$$

Thus, the unnormalised wavefunctions for this potential are

$$\psi_n(y;\,A,\,B) \propto (1-y)^{B/2\alpha}(1+y)^{A/2\alpha}P_n^{(B/\alpha-\frac{1}{2},\,A/\alpha-\frac{1}{2})}(y). \tag{21}$$

Table 1. All known shape-invariant potentials with their properties including energy eigenvalues and eigenfunctions ($\hbar = 2m = 1$), n denotes the radial quantum number and α, A, $B \geq 0$. Unless stated otherwise, the range of the potentials is $-\infty < x < \infty$, $0 \leq r < \infty$. For spherically symmetric potentials the full wavefunction is $\psi_{nlm}(r) = (1/r)\psi_n(r)Y_{lm}(\theta, \phi)$. Notice that the wavefunctions for the first four potentials in table 1 are special cases of confluent hypergeometric while the rest are special cases of the hypergeometric function.

Name of potential	Superpotential $W(x)$	Potential $V_-(x; a_0)$	$\{a_0\}$	$\{a_1\}$	Eigenvalues E_n	Variable y	Wavefunction $\psi_n(y)$	Recursion relation
Shifted oscillator	$\frac{1}{2}\omega x - b$	$\frac{1}{4}\omega^2(x-2b/\omega)^2 - \frac{1}{2}\omega$	ω	ω	$n\omega$	$y = (\frac{1}{2}\omega)^{1/2}(x-2b/\omega)$	$\exp(-\frac{1}{2}y^2)H_n(y)$	(A1)
Three-dimensional oscillator	$\frac{1}{2}\omega r - \frac{(l+1)}{r}$	$\frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - (l+\frac{3}{2})\omega$	l	$l+1$	$2n\omega$	$y = \frac{1}{2}\omega r^2$	$\exp(-\frac{1}{2}y)y^{(l+1)/2}L_n^{l+1/2}(y)$	(A2b)
Coulomb	$\frac{e^2}{2(l+1)} - \frac{(l+1)}{r}$	$-\frac{e^2}{r} + \frac{l(l+1)}{r^2} - \frac{e^4}{4(l+1)^2}$	l	$l+1$	$\frac{e^4}{4}\left(\frac{1}{(l+1)^2} - \frac{1}{(n+l+1)^2}\right)$	$y = \frac{re^2}{(n+l+1)}$	$y^{l+1}\exp(-\frac{1}{2}y)L_n^{2l+1}(y)$	(A2c)
Morse	$A - B\exp(-\alpha x)$	$A^2 + B^2\exp(-2\alpha x)$ $-2B(A+\alpha)\exp(-\alpha x)$	A	$A-\alpha$	$A^2 - (A-n\alpha)^2$	$y = (2B/\alpha)\exp(-\alpha x)$, $s = A/\alpha$	$y^{s-n}\exp(-\frac{1}{2}y)L_n^{2s-2n}(y)$	(A2a)
	$A\tanh\alpha x + B\,\mathrm{sech}\,\alpha x$	$A^2 + (B^2 - A^2 - A\alpha)\,\mathrm{sech}^2\,\alpha x$ $+ B(2A+\alpha)\,\mathrm{sech}\,\alpha x\tanh\alpha x$	A	$A-\alpha$	$A^2 - (A-n\alpha)^2$	$y = \sinh\alpha x$, $s = A/\alpha$, $\lambda = B/\alpha$	$i^n(1+y^2)^{-s/2}\exp(-\lambda\tan^{-1}y)$ $\times P_n^{(i\lambda-s-\frac{1}{2}, -i\lambda-s-\frac{1}{2})}(iy)$	(A3b)
Rosen–Morse	$A\tanh\alpha x + B/A$	$A^2 + B^2/A^2 + 2B\tanh\alpha x$ $-A(A+\alpha)\,\mathrm{sech}^2\,\alpha x$	A	$A-\alpha$	$A^2 - (A-n\alpha)^2$ $+B^2/A^2 - B^2/(A-n\alpha)^2$	$y = \tanh\alpha x$, $s = A/\alpha$, $\lambda = B/\alpha^2$, $a = \lambda/(s-n)$	$(1-y)^{(s-n+a)/2}(1+y)^{(s-n-a)/2}$ $\times P_n^{(s-n+a, s-n-a)}(y)$	(A3a)
	$A\coth\alpha r - B\,\mathrm{cosech}\,\alpha r$ $(A < B)$	A^2 $+(B^2 + A^2 + A\alpha)\,\mathrm{cosech}^2\,\alpha r$ $-B(2A+\alpha)\coth\alpha r\,\mathrm{cosech}\,\alpha r$	A	$A-\alpha$	$A^2 - (A-n\alpha)^2$	$y = \cosh\alpha r$, $s = A/\alpha$, $\lambda = B/\alpha$	$(y-1)^{(\lambda-s)/2}(y+1)^{-(\lambda+s)/2}$ $\times P_n^{(\lambda-s-\frac{1}{2}, -\lambda-s-\frac{1}{2})}(y)$	(A3b)
Eckart	$-A\coth\alpha r + B/A$ $(B > A^2)$	$A^2 + B^2/A^2 - 2B\coth\alpha r$ $+A(A-\alpha)\,\mathrm{cosech}^2\,\alpha r$	A	$A+\alpha$	$A^2 - (A+n\alpha)^2 + B^2/A^2$ $-B^2/(A+n\alpha)^2$	$y = \coth\alpha r$, $s = A/\alpha$, $\lambda = B/\alpha^2$, $a = \lambda/(n+s)$	$(y-1)^{-(s+n-a)/2}(y+1)^{-(s+n+a)/2}$ $\times P_n^{(-s-n+a, -s-n-a)}(y)$	(A3a)
	$-A\cot\alpha x$ $+B\,\mathrm{cosec}\,\alpha x$ $(0 \leq \alpha x \leq \pi;\ A > B)$	$-A^2$ $+(A^2 + B^2 - A\alpha)\,\mathrm{cosec}^2\,\alpha x$ $-B(2A-\alpha)\cot\alpha x\,\mathrm{cosec}\,\alpha x$	A	$A+\alpha$	$(A+n\alpha)^2 - A^2$	$y = \cos\alpha x$, $s = A/\alpha$, $\lambda = B/\alpha$	$(1-y)^{(s-\lambda)/2}(1+y)^{(s+\lambda)/2}$ $\times P_n^{(s-\lambda-\frac{1}{2}, s+\lambda-\frac{1}{2})}(y)$	(A3b)
Pöschl–Teller I	$A\tan\alpha x$ $-B\cot\alpha x$ $(0 \leq \alpha x \leq \frac{1}{2}\pi)$	$-(A+B)^2$ $+A(A-\alpha)\sec^2\alpha x$ $+B(B-\alpha)\,\mathrm{cosec}^2\,\alpha x$	(A, B)	$(A+\alpha, B+\alpha)$	$(A+B+2n\alpha)^2$ $-(A+B)^2$	$y = 1 - 2\sin^2\alpha x$, $s = A/\alpha$, $\lambda = B/\alpha$	$(1-y)^{\lambda/2}(1+y)^{s/2}$ $\times P_n^{(\lambda-\frac{1}{2}, s-\frac{1}{2})}(y)$	(A3b)
Pöschl–Teller II	$A\tanh\alpha r$ $-B\coth\alpha r$ $(B < A)$	$(A-B)^2$ $-A(A+\alpha)\,\mathrm{sech}^2\,\alpha r$ $+B(B-\alpha)\,\mathrm{cosech}^2\,\alpha r$	(A, B)	$(A-\alpha, B+\alpha)$	$(A-B)^2$ $-(A-B-2n\alpha)^2$	$y = -1 + 2\sinh^2\alpha r$, $s = A/\alpha$, $\lambda = B/\alpha$	$(y-1)^{\lambda/2}(y+1)^{-s/2}$ $\times P_n^{(\lambda-\frac{1}{2}, -s-\frac{1}{2})}(y)$	(A3b)

The procedure outlined above has been applied to all known shape-invariant potentials and the final results are summarised in table 1. In this table we have given the superpotential $W(x)$, potential $V_-(x)$, values of the parameters $\{a_0\}$, $\{a_1\}$, bound-state energy E_n, energy eigenfunctions $\psi_n(y)$ and the variable y, as well as the recursion relation used.

Special mention may be made of the potential for which $W(x) = A\tanh\alpha x + B\,\mathrm{sech}\,\alpha x$. As far as we are aware energy eigenfunctions have not been explicitly worked out in the literature for this potential. Gendenshtein (1983) obtained its energy eigenvalues by using shape invariance. From the table it would appear that the energy eigenfunctions for this potential are complex. However, this is not so. We have explicitly computed the first few eigenfunctions and have checked that they are all real and orthonormal. In particular, the first three unnormalised energy eigenfunctions are ($y = \sinh\alpha x$, $A = s\alpha$, $B = \lambda\alpha$):

$$\psi_0(y, s) = (1+y^2)^{-s/2}\exp[-\lambda\tan^{-1}y] \tag{22}$$

$$\psi_1(y, s) = \psi_0(y, s)[2\lambda + (2s-1)y] \tag{23}$$

$$\psi_2(y, s) = \psi_0(y, s)\{[(2s-1)y+2\lambda][(2s-3)y+2\lambda] - (2s-3)(1+y^2)\}. \tag{24}$$

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Appendix

The following recursion relations for various orthogonal polynomials have been used in deriving the wavefunctions $\psi_n(x)$. They can readily be obtained by manipulating recursion relations found in standard books (Abramowitz and Stegun 1970).

Hermite:

$$H_n(y) = 2yH_{n-1}(y) - \frac{d}{dy}H_{n-1}(y). \tag{A1}$$

Laguerre:

$$nL_n^\alpha = y\frac{d}{dy}L_{n-1}^\alpha + [(\alpha+n) - y]L_{n-1}^\alpha \tag{A2a}$$

$$nL_n^{\alpha-1} = y\frac{d}{dy}L_{n-1}^\alpha + [\alpha - y]L_{n-1}^\alpha \tag{A2b}$$

$$n(n-1+\alpha)L_n^{\alpha-2} = (\alpha-1)y\frac{d}{dy}L_{n-1}^\alpha + [\alpha(\alpha-1) - y(\alpha+n-1)]L_{n-1}^\alpha. \tag{A2c}$$

Jacobi:

$$\begin{aligned}2n(\alpha+\beta+n)p_n^{(\alpha,\beta)} &= (\alpha+\beta+n)[(\alpha-\beta)+(\alpha+\beta+2n)y]P_{n-1}^{(\alpha,\beta)}\\ &\quad -(\alpha+\beta+2n)(1-y^2)\frac{d}{dy}P_{n-1}^{(\alpha,\beta)}\end{aligned} \tag{A3a}$$

$$2nP_n^{(\alpha,\beta)} = [(\alpha-\beta)+(\alpha+\beta+2)y]P_{n-1}^{(\alpha+1,\beta+1)} - (1-y^2)\frac{d}{dy}P_{n-1}^{(\alpha+1,\beta+1)}. \tag{A3b}$$

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